

Domination polynomials of k -tree related graphs

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ABSTRACT

Let G be a simple graph of order n . The domination polynomial of G is the polynomial $D(G, x) = \sum_{i=\gamma(G)}^n d(G, i)x^i$, where $d(G, i)$ is the number of dominating sets of G of size i and $\gamma(G)$ is the domination number of G . In this paper we study the domination polynomials of several classes of k -tree related graphs. Also, we present families of these kind of graphs, whose domination polynomial have no nonzero real roots.

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1 Introduction

Throughout this paper we will consider only simple graphs. Let $G = (V, E)$ be a simple graph. For $F \subseteq V(G)$ we use $\langle F \rangle$ for the subgraph induced by F . For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) | \{u, v\} \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. For every vertex $v \in V(G)$, the *degree* of v is the number of edges incident with v and is denoted by $d_G(v) = |N(v)|$. Let d_i , $1 \leq i \leq n$, be the degrees of the vertices v_i of a graph in any order. The sequence $\{d_i\}_1^n$ is called the *degree sequence* of the graph. A *clique* in a graph is a subset of its vertices such that every two vertices in the subset are connected by an edge. We use K_n, P_n, C_n and $S_{1,n-1}$ for a clique, a path, a cycle and a star, all of order n , respectively.

A set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V$ or equivalently, every vertex in $V(G) \setminus S$ is

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adjacent to at least one vertex in S . The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G . A dominating set with cardinality $\gamma(G)$ is called a γ -set. For a detailed treatment of these parameters, the reader is referred to [13]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G, i) = |\mathcal{D}(G, i)|$. The *domination polynomial* $D(G, x)$ of G is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i)x^i$, where $\gamma(G)$ is the domination number of G (see [2, 7]). Thus $D(G, x)$ is the generating polynomial for the number of dominating sets of G of each cardinality. A root of $D(G, x)$ is called a *domination root* of G .

In [14] it is shown that computing the domination polynomial $D(G, x)$ of a graph G is NP-hard and some examples for graphs for which $D(G, x)$ can be computed efficiently are given. The vertex contraction G/u of a graph G by a vertex u is the operation under which all vertices in $N(u)$ are joined to each other and then u is deleted (see [17]). The following theorem is useful for finding the recurrence relations for the domination polynomials of arbitrary graphs.

Theorem 1.[5, 14] *Let G be a graph. For any vertex u in G we have*

$$D(G, x) = xD(G/u, x) + D(G - u, x) + xD(G - N[u], x) - (1 + x)p_u(G, x),$$

where $p_u(G, x)$ is the polynomial counting the dominating sets of $G - u$ which do not contain any vertex of $N(u)$ in G .

Using Theorem 1 we are able to obtain an easier formula for a graph with at least one vertex of degree 1. Since every tree has at least two vertices of degree 1, so we can use the following recurrence to obtain the domination polynomials of trees.

Corollary 1.[8, 14] *Let $G = (V, E)$ be a graph, v be a vertex of degree 1 in G and let u be its neighbor. Then*

$$D(G, x) = xD(G/u, x) + D(G - u - v, x) + D(G - N[u], x).$$

If G_1 and G_2 are disjoint graphs of orders n_1 and n_2 respectively, then $D(G_1 \cup G_2, x) = D(G_1, x)D(G_2, x)$ and

$$D(G_1 + G_2, x) = \left((1 + x)^{n_1} - 1\right)\left((1 + x)^{n_2} - 1\right) + D(G_1, x) + D(G_2, x),$$

where $G_1 + G_2$ is the join of G_1 and G_2 , formed from $G_1 \cup G_2$ by adding in all edges between a vertex of G_1 and a vertex of G_2 (see [2]).

The domination polynomials of trees, aside from path graph have not been studied and there is no study for coefficients of $D(T, x)$ for trees T with n vertices. k -trees are generalization of tree which consider in this paper. Actually similar to [16], in this paper we consider k -tree related graphs and study their domination polynomials. Study of the roots of domination polynomials is a interesting ([1, 10]). One of the problem in domination roots is classification and finding graphs with no nonzero real roots. In this paper we present some families related to k -trees which have this property.

In Section 2, we study the domination polynomials for some k -tree related graphs. In Section 3, we present some families of these kind of graphs whose domination polynomials have no nonzero real roots.

2 Domination polynomials of k -tree related graphs

In this section we study the domination polynomials for some k -tree related graphs. The class of k -trees is a very important subclass of triangulated graphs. Harary and Palmer [12] first introduced 2-trees in 1968. Beineke and Pippert [9] gave the definition of a k -tree in 1969. In the literature on k -trees, there are interesting applications to the study of computational complexity.

Definition 1. For a positive integer k , a k -tree, denoted by T_n^k , is defined recursively as follows: The smallest k -tree is the k -clique K_k . If G is a k -tree with $n \geq k$ vertices and a new vertex v of degree k is added and joined to the vertices of a k -clique in G , then the larger graph is a k -tree with $n + 1$ vertices.

An *independent set* in a graph G is a set of pairwise non-adjacent vertices.

Definition 2. Let K_k be a k -clique and S be an independent set of $n - k$ vertices. A (k, n) -star, denoted by $S_{k, n-k}$, is defined as $S_{k, n-k} = K_k + S$.

Definition 3. A (k, n) -path, denoted by P_n^k , begins with k -clique on $\{v_1, v_2, \dots, v_k\}$. For $i = k + 1$ to n , let vertex v_i be adjacent to vertices $\{v_{i-1}, v_{i-2}, \dots, v_{i-k}\}$ only. (see Figure 1).

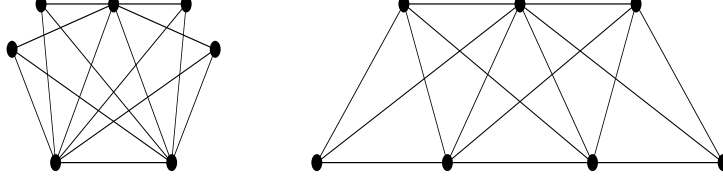


Figure 1: The 3-star and 3-path on 7 vertices, respectively.

A helpful characteristic of the (k, n) -path P_n^k is that we may order the vertices v_1, v_2, \dots, v_n such that $P_n^k \setminus \{v_1, \dots, v_i\}$ is a k -path on $n - i$ vertices for $1 \leq i \leq n - k - 1$, such a vertex ordering is referred to as a presentation.

Definition 4. A (k, n) -cycle, denoted by C_n^k , consists of a (k, n) -path on $\{v_1, v_2, \dots, v_n\}$ defined as above and an edge joining v_1 to v_n , where $n \geq k + 2$.

Definition 5. If G is a (k, n) -cycle of order n and v is a vertex not in G , then $G + v$ is called a (k, n) -wheel and denoted by W_n^k .

Notice that P_n^1 , C_n^1 , W_n^1 and $S_{1, n-1}$ are just the standard path, cycle, wheel and star, respectively. It follows easily from the domination polynomial of join of two graphs that, for the star graph $K_{1, n-1}$, we have $D(K_{1, n}, x) = x(1 + x)^{n-1} + x^{n-1}$. The following are recurrences for the domination polynomials of paths and cycles ([6]).

Theorem 2. For the natural number $n \geq 3$,

$$(i) \quad D(P_{n+1}, x) = x(D(P_n, x) + D(P_{n-1}, x) + D(P_{n-2}, x)),$$

$$\text{where } D(P_1, x) = x, \quad D(P_2, x) = x^2 + 2x \text{ and } D(P_3, x) = x^3 + 3x^2 + x.$$

$$(ii) \quad D(C_{n+1}, x) = x(D(C_n, x) + D(C_{n-1}, x) + D(C_{n-2}, x)),$$

$$\text{where } D(C_1, x) = x, \quad D(C_2, x) = x^2 + 2x \text{ and } D(C_3, x) = x^3 + 3x^2 + 3x.$$

Note that both (k, n) -cycles and (k, n) -wheels are not k -trees. But they are closely related to k -trees. We begin by a simple lemma was proven in [16] as Proposition 2:

Lemma 1. For any k -tree T_n^k , $|E(T_n^k)| = kn - \frac{1}{2}k(k+1) = \frac{2nk-k^2-k}{2}$.

The *independence number* is the size of a maximum independent set in the graph and denoted by $\alpha(G)$. The following Lemma gives independence numbers for k -tree related graphs:

Lemma 2.[16] For each natural number $k \leq n$, we have

- (i) $\alpha(P_n^k) = \lfloor \frac{n+k}{k+1} \rfloor$.
- (ii) $\alpha(C_n^k) = \lfloor \frac{n+k-1}{k+1} \rfloor$.
- (iii) $\alpha(W_n^k) = \alpha(C_n^k)$.
- (iv) $\alpha(S_{k,n-k}) = n - k$.

Now, we present the following domination numbers for k -tree related graphs:

Theorem 3. For each natural number $k \leq n$, we have

- (i) $\gamma(P_n^k) = \lceil \frac{n}{2k+1} \rceil$.
- (ii) $\gamma(C_n^k) = \lceil \frac{n}{2k+1} \rceil$.
- (iii) $\gamma(W_n^k) = 1$.
- (iv) $\gamma(S_{k,n-k}) = 1$.

Proof. (i) Since $k \leq n$, by the definition of P_n^k , the degree sequence in this graph is

$$\{k, k+1, \dots, 2k-1, 2k, \dots, 2k, 2k-1, \dots, k-1, k\},$$

we have $\gamma(P_n^k) = 1$ for $n \leq 2k+1$ and $\gamma(P_n^k) = 2$ for $2k+2 \leq n \leq 4k+2$. Thus (i) holds for $k \leq n \leq 4k+2$. Now assume $n \geq 4k+3$. We use induction on n . Since any γ -set of P_n^k contains only one vertex of the $\{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ and $P_n^k - \{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ is a k -path with $n-2k-1$ vertices, by induction, $\gamma(P_n^k) = 1 + \gamma(P_{n-2k-1}^k) = 1 + \lceil \frac{n-2k-1}{2k+1} \rceil = \lceil \frac{n}{2k+1} \rceil$. Hence (i) holds.

(ii) Since $k \leq n$, by the definition of C_n^k , the degree sequence in this graph is

$$\{k+1, k+1, k+2, \dots, 2k-1, 2k, \dots, 2k, 2k-1, \dots, k-1, k+1\},$$

we have $\gamma(C_n^k) = 1$ for $n \leq 2k+1$ and $\gamma(C_n^k) = 2$ for $2k+2 \leq n \leq 4k+2$. Thus (ii) holds for $k \leq n \leq 4k+2$. Now assume $n \geq 4k+3$ and use induction on n . Since any γ -set of C_n^k contains only one vertex of the $\{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ and $C_n^k - \{v_n, v_{n-1}, \dots, v_{n-k}, \dots, v_{n-2k-1}\}$ is a k -path with $n-2k-1$ vertices, by induction, $\gamma(C_n^k) = 1 + \gamma(P_{n-2k-1}^k) = 1 + \lceil \frac{n-2k-1}{2k+1} \rceil = \lceil \frac{n}{2k+1} \rceil$. Hence (ii) holds.

(iii) Since the (k, n) -wheel W_n^k , has a vertex v of degree $n-1$, so (iii) holds.

(iv) Since the k -star graph $S_{k, n-k}$, has k vertices of degree $n-1$, so (iv) holds. \square

The following theorem gives a recurrence formula for the domination polynomial of (k, n) -path graphs.

Theorem 4. *If $n \leq k+1$, then $D(P_n^k, x) = D(K_n, x)$. For every $k+2 \leq n$,*

$$D(P_n^k, x) = (1+x)D(P_{n-1}^k, x) + xD(P_{n-k-1}^k, x) - (1+x)p_u(P_n^k, x),$$

$$\text{where } p_u(P_n^k, x) = \begin{cases} x(1+x)^{n-k-2}, & k+2 \leq n \leq 2k+2, \\ x((1+x)^{n-k-2} - (1+x)^{n-2k-3}); & 2k+3 \leq n \leq 2k+6, \\ p_u(P_n^k, x); & 2k+7 \leq n. \end{cases}$$

Proof. If $n \leq k+1$, then $P_n^k \cong K_n$. For every $k+2 \leq n$, we use Theorem 1 for the last vertex of P_n^k and since (by the definition of P_n^k) the first $k+1$ and the last $k+1$ vertices form two cliques, we have $P_n^k/u \cong P_n^k - u$. It is clear that $D(P_n^k - N[u], x) = D(P_{n-k-1}^k, x)$. Obviously $p_u(P_n^k, x)$ is the polynomial counting the dominating sets of P_{n-k-1}^k contains the vertex v_{n-k-1} , but finding this polynomial involve complex calculations. We brought this polynomial for $n \leq 2k+6$ in this theorem. Therefore we have the result. \square

In general, finding the domination polynomial of a graph is a very difficult problem. In [14] Kotek et al. showed that there exist recurrence relations for the domination polynomial which allow for efficient schemes to compute the polynomial for some types of graphs. Consider (k, n) -cycle graphs, If $n \leq k+2$, then $C_n^k \cong K_n$. Consequently in this case $D(C_n^k, x) = D(K_n, x)$. For every $k+3 \leq n$, until now all attempts to find formulas for $D(C_n^k, x)$ failed.

The following theorem gives a formula for the domination polynomial of (k, n) -wheel graphs.

Theorem 5. *For a (k, n) -wheel W_n^k , we have*

$$D(W_n^k, x) = x(1+x)^{n-1} + D(C_n^k, x).$$

Proof. Since $W_n^k = C_n^k + K_1$, then

$$\begin{aligned} D(W_n^k, x) &= ((1+x) - 1)((1+x)^{n-1} - 1) + x + D(C_n^k, x) \\ &= x(1+x)^{n-1} + D(C_n^k, x). \quad \square \end{aligned}$$

The following theorem gives a formula for the domination polynomial of k -star graphs, which is concluded of the fact, k -star graph is the join of complete graph K_k and independent set S (empty graph O_{n-k}).

Theorem 6. *For every $k \in \mathbb{N}$ and $n > k$,*

$$D(S_{k,n-k}, x) = (1+x)^{n-k}((1+x)^k - 1) + x^{n-k}.$$

Proof. Let $S_{k,n-k}$ be the k -star graph with vertex set $V(S_{k,n-k}) = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$. It suffices to show that every dominating set of size $j = 1, \dots, n$ is accounted for exactly once in the above statement. Clearly every non-empty subset of $\{v_1, v_2, \dots, v_k\}$ is dominating set of k -star graphs. These sets can be extended with any number of vertices $\{v_{k+1}, \dots, v_n\}$. Also obviously the set $\{v_{k+1}, \dots, v_n\}$ is a dominating set of k -star graphs. It is easy to see that there is no another method to make a dominating set for k -star graphs. Therefore we have the result. \square

The value of a graph polynomial at a specific point can give sometimes a surprising information about the structure of the graph [4, 15]. The following simple results give the domination polynomial of k -tree related graphs at -1 .

Corollary 2. *For each natural number $k \leq n$, the following hold:*

$$(i) \ D(P_n^k, -1) = (-1)^{\alpha(P_n^k)}.$$

- (ii) $D(W_n^k, -1) = D(C_n^k, -1)$.
 (iii) $D(S_{k,n-k,-1}) = (-1)^{\alpha(S_{k,n-k})}$.

Proof. (i) Using domination polynomial of (k, n) -path in Theorem 4, for $k \leq n \leq k+1$, $D(P_n^k, -1) = D(K_n, -1) = -1$. For every $k+2 \leq n$, $D(P_n^k, -1) = -D(P_{n-k-1}^k, -1)$. Obviously, in the first case $\lfloor \frac{n+k}{k+1} \rfloor = 1$. Thus (i) holds for $n \leq k+1$. Now assume $k+2 \leq n$. We use induction on n . Suppose that the statement is true for every k -path with $n-k$ vertices, by induction and Lemma 2,

$$\begin{aligned} D(P_n^k, -1) &= -D(P_{n-k-1}^k, -1) \\ &= -(-1)^{\alpha(P_{n-k-1}^k)} = -(-1)^{\lfloor \frac{n-1}{k+1} \rfloor} \\ &= (-1)^{1+\lfloor \frac{n-1}{k+1} \rfloor} = (-1)^{\lfloor \frac{n+k}{k+1} \rfloor}. \end{aligned}$$

Hence (i) holds.

(ii) Follows from Theorem 5.

(iii) Follows from Theorem 6 and Lemma 2. \square

3 Some families of graphs with no nonzero real domination roots

In [1] authors asked that which graphs have no nonzero real domination roots?

In this section we would like to obtain more results related to this problem. We need some preliminaries.

For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of G with $|V|$ copies of H , by adding edges between the i th vertex of G and all vertices of i th copy of H [11]. It is easy to see that the corona operation of two graphs does not have the commutative property.

We need the following theorem which is for computation of domination polynomial of corona products of two graphs.

Theorem 7.[3, 14] *Let $G = (V, E)$ and $H = (W, F)$ be nonempty graphs of order n and m ,*

respectively. Then

$$D(G \circ H, x) = (x(1+x)^m + D(H, x))^n.$$

A k -star, $S_{k,n-k}$, has vertex set $\{v_1, \dots, v_n\}$ where $\langle \{v_1, v_2, \dots, v_k\} \rangle \cong K_k$ and $N(v_i) = \{v_1, \dots, v_k\}$ for $k+1 \leq i \leq n$.

Here we will discuss roots of domination polynomial of k -star graphs.

Theorem 8.

- (i) For odd natural n and even natural k , no nonzero real numbers is domination root of $S_{k,n-k}$.
- (ii) For even natural n and even natural k , there is exactly one nonzero real domination root of $S_{k,n-k}$.

Proof. By Theorem 6, for every $n > k$, $D(S_{k,n-k}, x) = (1+x)^{n-k}((1+x)^k - 1) + x^{n-k}$. If $D(S_{k,n-k}, x) = 0$, then we have

$$(1+x)^{n-k}((1+x)^k - 1) = -x^{n-k}.$$

Now we are ready to prove two cases of this theorem:

- (i) First suppose that $x \geq 0$. Obviously the above equality is true just for real number 0, since for nonzero real number the left side of equality is positive but the right side is negative. Now suppose that $x < -1$. In this case the left side is negative and the right side $-x^{n-k}$ is greater than +1, a contradiction. Finally we shall consider $-1 < x < 0$. This case is similar to the second case when we substitute x with $\frac{1}{x}$.
- (ii) First suppose that $x \geq 0$ and $x < -1$. Obviously the above equality is true just for real number 0, since for nonzero real number the left side of equality is positive but the right side is negative. Now suppose that $-1 < x < 0$. This equation has only one real root in $(-1, 0)$. \square

Remark. Using Maple we have shown the domination roots of $S_{4,n-4}$ for $5 \leq n \leq 44$ in Figure 2.

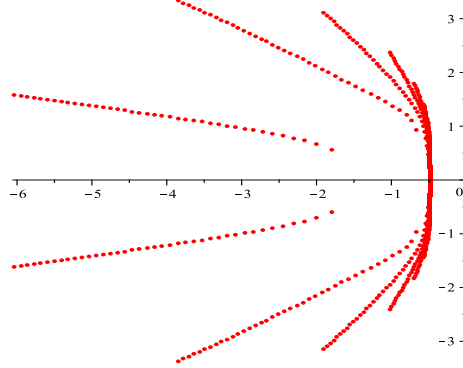


Figure 2: Domination roots of graphs $S_{4,n-4}$ for $5 \leq n \leq 44$.

Here we construct a sequence of graphs, which their domination roots are the same as the domination roots of the $k + 1$ -star graphs.

Theorem 9. *The domination roots of every graph H in the family*

$$\{G \circ S_{k,n-k}, (G \circ S_{k,n-k}) \circ S_{k,n-k}, ((G \circ S_{k,n-k}) \circ S_{k,n-k}) \circ S_{k,n-k}, \dots\}$$

have the same behavior as the domination roots of $k + 1$ -star graphs.

Proof. By theorem 7 we can deduce that for each arbitrary graph G ,

$$\begin{aligned} D(G \circ S_{k,n-k}, x) &= \left(x(1+x)^n + (1+x)^{n-k}((1+x)^k - 1) + x^{n-k} \right)^{|V(G)|} \\ &= \left((1+x)^{n-k}((1+x)^{k+1} - 1) + x^{n-k} \right)^{|V(G)|} \\ &= \left(D(S_{k+1,n-k}, x) \right)^{|V(G)|}. \end{aligned}$$

Therefore we have the result. \square

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